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Jensen's Alpha in the CAPM with Heterogeneous Beliefs*

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Abstract

Jensen's alpha is one of the most used terms in finance. Yet, the alpha is “mystical” since it has no theory. It is, for example, in contradiction to the standard CAPM with homogeneous beliefs. The purpose of this paper is to show that the alpha naturally arises in a financial market equilibrium when the CAPM is extended to heterogeneous beliefs. We show that the hunt for alpha opportunities is a zero-sum game and that alpha opportunities erode with the assets under management. Moreover, it is shown that a positive alpha is not necessarily a good criterion for the choice between active and passive investment. Finally, we argue that the standard CAPM with homogeneous beliefs can be seen as the long run outcome of our model when investors' expectations are linked to the trading success.

Keywords: CAPM, heterogeneous beliefs, active and passive investment.

JEL-Classification: G11, G12, G14.

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1 Introduction

The Capital Asset Pricing Model, CAPM, was the first general equilibrium model of a financial market and it is still the basis for many practical financial decisions. Its asset pricing implication is the security market line, SML, according to which the excess return of any asset over the risk free rate is proportional to the excess return of the market portfolio over the risk free rate. The proportionality factor is the *beta*, i.e. the covariance of the asset's return to the return of the market portfolio divided by the variance of the market portfolio. The beta is the only risk factor that is rewarded according the CAPM. Hence, investors requiring a high expected return will have to accept a high beta. Some investors, however, want to achieve more. They claim to be able to achieve positive deviations of expected returns over those given by the SML. Those deviations of returns are referred to as Jensen's alpha (Jensen, 1969) or short as the "*alpha*." Indeed the alpha is nowadays a common term in the finance jargon. Hedge funds, for example, consider themselves to be alpha generating strategies; many of them use the term "alpha" in their marketing brochures and some of them even as part of their name.¹

While many opinion leaders in the world of finance claim that the existence of alpha contradicts the validity of the CAPM, we argue in this paper that a simple extension of the CAPM towards heterogenous beliefs is already able to explain the alpha in a financial market equilibrium. The extension we use goes back to the CAPM with heterogenous beliefs suggested by Lintner (1969). For most of our results it will be sufficient to consider a CAPM, where investors have heterogenous beliefs over expected returns while they agree on the covariances of returns. The assumption of homogeneous covariance expectations is frequently used in the literature² and can be justified as many practitioners do portfolio allocations using historic covariances while adjusting historic means to get reasonable expected returns. Finally, in the CAPM means are first order effects

¹To list some examples: Goldman Sachs offers "Global Alpha", Merrill Lynch "Absolute Alpha Fund" and UBS "Alpha Hedge" and "Alpha Select".

²The famous model of Brock and Hommes (1998), for example, is based on mean-variance optimizing agents that have heterogenous beliefs on expected returns but agree on covariances. This model does, however, only have one risky asset.

while covariances capture second order effects. Hence, mistakes in means hurt the investor more than equally sized mistakes in covariances (cf. Chopra and Ziemba, 1993).

Our first result parallels Chiarella, Dieci, and He (2006) and derives the security market line of the CAPM as an aggregation result without using the unrealistic two-fund-separation property. The security market line turns out to hold with respect to *average* beliefs about the expected asset returns and covariances of returns. However, under heterogeneous expectations this security market line does not coincide with the individual security market lines defined with respect to investors' subjective beliefs. Hence, unlike in the CAPM with homogeneous beliefs, investors in equilibrium will hold different portfolios of risky assets. In particular, the often observed feature of underdiversification (see, for example, Odean, 1999; Goetzmann and Kumar, 2008, and Polkovnichenko, 2005) can well be compatible with equilibrium. If an investor has superior information, then underdiversification can even be necessary to outperform the market.

In our model alpha opportunities can be explained as a feature of financial market equilibria. The further the average expected returns deviate from the true returns the higher the alpha opportunities. Moreover we can show that alpha opportunities erode with the assets under management, which is a feature that has been observed for many active portfolio managers, as for example for hedge funds (cf. Getmansky, 2004, and Agarwal, Daniel, and Naik, 2007). In our model this important feature has a very simple explanation. The more wealth a strategy acquires the more it resembles the market portfolio, which, by definition, has an alpha of zero. Note that our model gives an equilibrium explanation of this feature that does not need to refer to any ad-hoc ideas of a production function for alpha opportunities (cf. Berk and Green, 2004). Moreover, in our model the hunt for alpha opportunities is a zero-sum game. If some investor generates a positive alpha there must be some other investor earning a negative alpha. Hence, the ease to generate alpha opportunities depends on the sophistication of the other investors in the market. This feature may explain why hedge funds could generate very high returns during the stock market bubble of the turn at the millennium in which many unsophisticated investors took active bets. After the bubble burst, many unsophisticated investors left the market due to frustration and hedge fund

returns decreased.

We extend our model by endogenizing the agents' information by allowing them to be either passive, in which case they invest according to the average expectation embodied in the market returns, or to be active, in which case they can acquire information at some cost. In our model we show that the decision of being active or passive depends on the efficiency of the market, the quality of the investor's belief, his degree of risk aversion and of course the costs for being active. An investor is more inclined to be active the less efficient the market is, the better his information and the less risk averse he is. By contrast, it can be shown that expecting a positive alpha is not necessarily a good criterion for becoming active. We give simple examples pointing out that expecting a positive alpha from the active strategy is neither a necessary nor a sufficient condition for becoming active. In our model, delegating active investment to portfolio managers only makes sense if the performance fee increases with the skill of the portfolio manager and is bounded above by some function of the degree of inefficiency of the market. Our model provides new measures for both of these components. Finally, in our model it turns out that a market in which some investors acquire information to be active while the others get the average information for free from market prices cannot be a stable outcome. Moreover, all investors being passive may also not be an outcome that is stable with respect to information acquisition if the average expectation is far from the true returns. This result resembles the well know result of Grossman and Stiglitz (1980) on the impossibility of informational efficient markets. Accounting for this stability requirement, the standard CAPM with homogenous and correct beliefs can be seen as the long run outcome of our model. Hence we can argue that alpha opportunities can arise in a financial market equilibrium as a reaction to a non-stationarity like an exogenous shock (invention of the railway, the mass production or the internet) but under sufficiently stationary exogenous conditions alpha opportunities will vanish.

Our results give a common framework for many phenomena that have been discussed in the literature. Besides being able to address alpha opportunities in a simple equilibrium framework, we can explain underdiversification, the erosion of alpha opportunities as assets under management increase, and the structure

of performance fees for active management. Moreover, our simple model gives a foundation of more applied research on active management like the one of Grinold and Kahn (2000) and Black and Litterman (cf. Litterman, 2003). Our model provides a common ground for these two approaches whose methodologies seem to be in contradiction. While Grinold and Kahn (2000) argue for active portfolio management based on the mean-variance framework of Markowitz, Black and Litterman argue for active portfolio management based on the security market line. Black and Litterman assume that the security market line is a “center of gravity” towards which the financial markets tend over time. Hence an active Black-Litterman investor goes short in those assets that have realized a positive alpha because he infers from this that in the next period the return will most likely be decreasing. Our model gives support to this view since taking account for the optimal information acquisition in the long run all alphas will erode. Our approach can also accommodate active portfolio management in the sense of Grinold and Kahn. As we show below, optimal mean-variance portfolios must lie on a security line which is the security market line in which market expectations have been replaced by individual expectations. The security market line is then obtained by the aggregation of these individual security lines. An active mean-variance investor à la Grinold and Kahn “sees” alpha opportunities because he holds a belief of expected returns that deviates from the average belief of the investors expressed in the security market line.

Of course we do not claim that our simple model can explain all features of active management. In particular some features related to hedge funds, as for example higher order returns, lead out of the mean-variance framework. However, since a simple CAPM with heterogenous beliefs carries us quite far in the understanding of many important features of active management this framework can give a first intuition for what active management is about.

The rest of the paper is organized as follows. Section 2 gives a formal description of the CAPM with heterogenous beliefs and it derives the aggregation result of the SML. Thereafter, in section 3, we consider the alpha. We prove the zero-sum property of alpha opportunities and their erosion if the assets under management increase. In section 4 we analyse the choice between active and passive portfolio management. We derive the main criterion for active portfo-

lio management based on the measures of market efficiency and the skill of the active managers. Furthermore we show which structure the fees for active management should have. Finally, we question whether expecting a positive alpha is an appropriate criterion for becoming active.

2 A CAPM with Heterogenous Beliefs

To be precise we need some notation. We consider a two periods financial market model with dates $t = 0, 1$, and K assets, $k = 0, 1, \dots, K$. Asset $k = 0$ is riskless and its return is denoted by R_f . Assets $k = 1, \dots, K$, are risky with asset prices denoted by q^k and asset returns by $R^k, k = 1, \dots, K$. We denote the exogenous supply of the risky assets by θ_k^M . The risk free asset has infinite elastic supply so that the risk free rate R_f can be considered as exogenously given. By $\hat{\mu}_k = \mathbb{E}(R^k)$ we denote the expected return of asset $k, k = 1, \dots, K$, and by $\text{COV} = (\text{COV}(R^k, R^l))_{k,l=1,\dots,K}$ we denote the covariance matrix of asset returns.

There are I investors. Investor i has initial wealth $w_0^i > 0$ and mean-variance preferences over date 1 returns

$$V^i(\mu, \sigma) = \mu - \frac{\gamma^i}{2} \sigma^2,$$

where $\gamma^i > 0$ measures investor i 's risk aversion and μ and σ are the expected return and variance, respectively, of investor i 's portfolio. We assume that investors do not know the distribution of asset payoffs but rather hold individual beliefs over expected asset returns and the covariance matrix of asset returns.

Let μ_k^i denote i 's belief about expected return of asset k and let COV^i denote i 's belief about the covariance matrix of returns.

Given her beliefs μ^i and COV^i investor i solves

$$\max_{\lambda \in \mathbb{R}^K} \lambda^T (\mu^i - R_f e) - \frac{\gamma^i}{2} \lambda^T \text{COV}^i \lambda, \quad (1)$$

where $e = (1, \dots, 1) \in \mathbb{R}^K$. The necessary and sufficient first order condition for

the solution λ^i of (1) is

$$\text{COV}^i \lambda^i = \frac{\mu^i - R_f e}{\gamma^i}. \quad (2)$$

Given the portfolio of risky assets λ^i investor i invests $\lambda_0^i = 1 - \sum_{k=1}^K \lambda_k^i$ into the riskless asset.

A *financial market equilibrium* then is a price vector q together with an allocation of optimal individual portfolios λ^i for which all markets are cleared, i.e. for all risky assets we have

$$q^k \theta_k^M = \sum_i w_0^i \lambda_k^i.$$

From now on we assume that in equilibrium $\sum_k \lambda_k^i > 0$, so that $\lambda_0^i < 1$ for all i . This assumption is not too restrictive but needed for the normalization we will do below.

2.1 Security Market Line

The asset pricing implication of the standard CAPM is the security market line (SML) according to which the excess return of any asset over the risk free rate is proportional to the excess return of the market portfolio over the risk free rate. The proportionality factor is the *beta*, i.e. the covariance of the asset's returns to the return of the market portfolio divided by the variance of the market portfolio. In order to derive the security market line for our heterogenous expectations we need to specify how individual expectations are averaged to become the market expectation.

Let $w_f^i := (1 - \lambda_0^i) w_0^i$ be the financial wealth investor i invests into risky assets. By our assumption above $w_f^i > 0$ for all i . Let, accordingly,

$$r^i = \frac{w_f^i}{\sum_j w_f^j},$$

be the relative financial wealth invested by i and define

$$\rho := \left[\sum_i \frac{r^i}{\gamma^i (1 - \lambda_0^i)} \right]^{-1}$$

In our model it turns out that the appropriate aggregation rule is to define the average belief about the expected asset returns, $\bar{\mu}$, and the average belief about the covariance matrix of asset returns, $\overline{\text{COV}}$ as follows:

$$\overline{\text{COV}} := \frac{1}{\rho} \left[\sum_i \frac{r^i}{\gamma^i(1 - \lambda_0^i)} (\text{COV}^i)^{-1} \right]^{-1} \quad (3)$$

$$\text{and } \bar{\mu} := \rho \overline{\text{COV}} \sum_i \frac{r^i}{\gamma^i(1 - \lambda_0^i)} (\text{COV}^i)^{-1} \mu^i. \quad (4)$$

$$(5)$$

Observe that under homogenous beliefs about the covariance matrix of asset returns, i.e. $\text{COV}^i = \text{COV}^*$ for all i , we obtain $\overline{\text{COV}} = \text{COV}^*$ and $\bar{\mu} = \sum_i a^i \mu^i$, where

$$a^i = \frac{w_0^i}{\gamma^i} \left(\sum_j \frac{w_0^j}{\gamma^j} \right)^{-1},$$

i.e. every individual's belief enters the average belief proportional to the individual's wealth divided by his risk aversion.

If all investors invest according to their risky portfolio $\bar{\lambda}^i$, then in equilibrium the relative market capitalizations of the risky assets or for short the “market portfolio” is

$$\lambda_k^M := \frac{q^k \theta_k^M}{\sum_{k=1}^K q^k \theta_k^M} = \sum_i r^i \bar{\lambda}_k^i, \quad k = 1, \dots, K,$$

where $\bar{\lambda}^i$ denotes i 's portfolio of risky assets, i.e.

$$\bar{\lambda}_k^i := \frac{\lambda_k^i}{1 - \lambda_0^i}, \quad k = 1, \dots, K.$$

Accordingly, let $\bar{\mu}^M = \sum_k \lambda_k^M \bar{\mu}_k$ be the average belief about the expected return $R^M = \sum_k \lambda_k^M R^k$ of the market portfolio. Then we can state the Security Market Line Theorem for average expectations as:

Proposition 2.1 (Security Market Line for Average Expectations)

In equilibrium the risk premium of any asset k is proportional to the risk premium of the market portfolio under average expectations. The factor of proportionality is given by the covariance of the return of asset k with the market portfolio

divided by the variance of the market portfolio, where covariances and variances are determined with respect to $\overline{\text{COV}}$:

$$\bar{\mu}_k - R_f = \underbrace{\frac{\overline{\text{COV}}(R^k, R^M)}{\bar{\sigma}^2(R^M)}}_{\bar{\beta}^{M,k}} (\bar{\mu}^M - R_f), \quad k = 1, \dots, K, \quad (6)$$

where $\overline{\text{COV}}(R^k, R^M) = \sum_l \lambda_l^M \overline{\text{COV}}_{k,l}$ and $\bar{\sigma}^2(R^M) = (\lambda^M)^T \overline{\text{COV}} \lambda^M$.

Proof: For all investors i we can rewrite (2) to obtain

$$\text{COV}^i \bar{\lambda}^i = \frac{\mu^i - R_f e}{\gamma^i (1 - \lambda_0^i)}. \quad (7)$$

From (7) it follows that

$$\begin{aligned} \lambda^M &= \sum_i r^i \bar{\lambda}^i = \sum_i \frac{r^i}{\gamma^i (1 - \lambda_0^i)} (\text{COV}^i)^{-1} (\mu^i - R_f e) \\ &= \frac{1}{\rho} \overline{\text{COV}}^{-1} (\bar{\mu} - R_f e) \end{aligned}$$

Hence,

$$\bar{\mu} - R_f e = \rho \overline{\text{COV}} \lambda^M \quad (8)$$

which implies that

$$\bar{\sigma}^2(R^M) = (\lambda^M)^T \overline{\text{COV}} \lambda^M = \frac{1}{\rho} (\bar{\mu}^M - R_f e) \quad (9)$$

Substituting (9) into (8) yields

$$\bar{\mu} - R_f e = \frac{\overline{\text{COV}} \lambda^M}{\bar{\sigma}_M^2} (\bar{\mu}^M - R_f e)$$

which proves the proposition. □

In the special case, where all investors have homogenous and correct beliefs about the covariance matrix of asset returns, i.e. $\text{COV}^i = \text{COV}$ for all i , the security market line for average expectations (6) reads

$$\bar{\mu}_k - R_f = \underbrace{\frac{\text{COV}(R^k, R^M)}{\sigma^2(R^M)}}_{\beta^{M,k}} (\bar{\mu}^M - R_f), \quad k = 1, \dots, K, \quad (10)$$

A similar aggregation result as in Proposition 2.1 has been obtained by Chiarella et al. (2006). While we consider a *distribution economy*, where there is an exogenously given income distribution among investors as well as an exogenously given supply of assets, Chiarella, Dieci, and He (2006) study an exchange economy, where investors are endowed with a portfolio of assets.³ Moreover, Chiarella et al. assume that investors have a linear mean-variance utility function over final wealth, while we assume that they have a linear mean-variance utility function over returns. This difference is crucial as it has implications for the comparative statics of portfolios with respect to wealth: In the model of Chiarella et al. the portfolios of risky assets held by the investors are independent of wealth, i.e. if wealth increases, then all additional wealth is invested into the riskless asset, which appears to be in conflict with observed investment behaviour. In contrast, the mean-variance utility function we consider yields a fixed mix portfolio, i.e. the share of wealth invested into a risky asset is independent of wealth. Taking investors' beliefs as given, Chiarella et al. focus on an analysis of the impact of the diversity of heterogenous beliefs on equilibrium prices and trading volume. In this paper we will go a step further and study which beliefs will survive in the long run. Hence, the degree of heterogeneity will be endogenous in our model.

Equation (6) is the security market line (SML) we obtain from aggregation of individual beliefs. This SML can be “seen” by an outside observer. An individual investor i , however, does not observe this SML. She sees an individual security market line defined with respect to her optimal portfolio of risky asset $\bar{\lambda}^i$ and her beliefs μ^i and COV^i . Let $R^{\bar{\lambda}^i} = \sum_l \bar{\lambda}_l^i R^l$ be the return of investor i 's portfolio of risky assets and let $\mu^i(R^{\bar{\lambda}^i}) = \sum_k \bar{\lambda}_k^i \mu_k^i$ be the expected return of her portfolio under her belief μ^i . Multiplying both sides of (7) with $\bar{\lambda}^i$ yields

$$\gamma^i(1 - \lambda_0^i) = \frac{\mu^i(R^{\bar{\lambda}^i}) - R_f}{\sigma^2(R^{\bar{\lambda}^i})},$$

where $\bar{\sigma}^2(R^{\bar{\lambda}^i}) = (\bar{\lambda}^i)^T \text{COV}^i \bar{\lambda}^i$. Substituting this into (7) we obtain the individual SML of investor i :

³For the difference between distribution and exchange economies cf. Malinvaud (1972).

Proposition 2.2 (Individual Security Market Line) *For any investor i the risk premium of any asset k is proportional to the risk premium of his portfolio, where the factor of proportionality is given by the covariance of the return of asset k with investor i 's portfolio divided by the variance of i 's portfolio and risk premia are determined according to μ^i :*

$$\mu_k^i - R_f = \underbrace{\frac{\text{COV}^i(R^k, R^{\bar{\lambda}^i})}{\sigma^{i2}(R^{\bar{\lambda}^i})}}_{\beta^{i,k}} \left(\mu^i(R^{\bar{\lambda}^i}) - R_f \right), \quad k = 1, \dots, K, \quad (11)$$

where $\text{COV}^i(R^k, R^{\bar{\lambda}^i}) = \sum_l \bar{\lambda}_l^i (\text{COV}^i)_{k,l}$ and $\sigma^{i2}(R^{\bar{\lambda}^i}) = (\bar{\lambda}^i)^T \text{COV}^i \bar{\lambda}^i$.

We see that investor i will hold the market portfolio if $\mu^i = \bar{\mu}$ and $\text{COV}^i = \overline{\text{COV}}$, i.e. if her beliefs coincide with the average beliefs in the market.

2.2 Underdiversification

There is considerable empirical evidence showing that the average investor is heavily underdiversified compared to the market portfolio (cf. Odean, 1999; Goetzmann and Kumar, 2008, and Polkovnichenko, 2005). In the CAPM with heterogeneous beliefs underdiversification is consistent with optimal investment. We illustrate this with the following simple example:

Example 2.1 Let there be two investors $i = 1, 2$, and two risky assets $k = 1, 2$. Assume that investors have a homogenous belief about the covariance matrix of asset returns, given by

$$\text{COV} = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix},$$

where $\sigma_1^2 > 0$ and $\sigma_2^2 > 0$. Moreover, assume that investor i 's belief about expected asset returns is given by

$$\mu^1 = \begin{pmatrix} d \\ R_f \end{pmatrix} \text{ and } \mu^2 = \begin{pmatrix} R_f \\ d \end{pmatrix},$$

where $d > R_f$. Then, it is straightforward to show that

$$\bar{\lambda}^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \bar{\lambda}^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence, investor 1 invests only into asset 1 and investor 2 only invests into asset 2, while the market portfolio is given by⁴

$$\lambda^M = \begin{pmatrix} r^1 \\ r^2 \end{pmatrix}.$$

Thus, in equilibrium each investor is underdiversified compared to the market portfolio.

3 The Alpha

The “alpha” is one of the most used terms in finance. It measures the deviation of mean asset returns from the security market line. Investment funds, in particular hedge funds, claim to generate a positive alpha in order to attract assets under management. As we will show in the following, our model of a CAPM with heterogenous beliefs can explain the existence of a nonzero alpha. However, at the same time our analysis will demonstrate that the alpha is not an appropriate performance measure when it comes to the choice between active and passive investment.

Under heterogenous beliefs there are several ways to define the alpha of a portfolio of risky assets as there are several security market lines. Any choice of beliefs and benchmark portfolios gives rise to a different security market line and hence to a different alpha. However, there are two alpha that appear to be more prominent than others. The first alpha is defined by the deviation from the individual security market line of an investor (11):

$$\alpha_k^i := \mu_k^i - R_f - \frac{\text{COV}^i(R^k, R^{\bar{\lambda}^i})}{\sigma^{i^2}(R^{\bar{\lambda}^i})} \left(\mu^i(R^{\bar{\lambda}^i}) - R_f \right), \quad k = 1, \dots, K.$$

Obviously, if investor i has chosen an optimal portfolio $\bar{\lambda}^i$, then $\alpha_k^i = 0$ for all k , i.e. any asset and hence all portfolios generate an alpha of zero given the beliefs of investor i . It turns out that α^i is the utility gradient of investor i in the optimal

⁴That is to say, we choose the exogenous supply of the risky assets, θ^M so that it matches the aggregate demand of the investors.

portfolio $\bar{\lambda}^i$. To see this, let $\alpha^{i,\lambda}$ be the utility gradient of investor i if she holds the (non-normalized) portfolio of risky assets $\lambda \in \mathbb{R}^K$, i.e.

$$\alpha^{i,\lambda} := \mu^i - R_f e - \gamma^i \text{COV}^i \lambda. \quad (12)$$

Let λ^{*i} be the optimal portfolio of investor i . Then $\alpha^i(\lambda^{*i}) = 0$ and we can solve for γ^i and substitute the expression back into (12). We obtain that for all $k = 1, \dots, K$,

$$\alpha_k^{i,\lambda} = \mu_k^i - R_f - \frac{\text{COV}^i \lambda^i}{\lambda^{*iT} \text{COV}^i \lambda^{*i}} \left(\lambda^{*iT} \mu^i - (1 - \lambda_0^{*i}) R_f \right) \quad (13)$$

$$= \mu_k^i - R_f - (1 - \lambda_0^{*i}) \frac{\text{COV}^i \lambda^i}{\sigma^2(R^{\bar{\lambda}^i})} \left(\mu^i(R^{\bar{\lambda}^i}) - R_f \right) \quad (14)$$

since $\bar{\lambda}_k^i = \lambda_k^{*i}/(1 - \lambda_0^{*i})$. If λ is chosen optimally, i.e. $\lambda = \lambda^{*i}$, then $\alpha^{i,\lambda^{*i}} = \alpha^i = 0$. Otherwise, the gradient $\alpha^{i,\lambda}$ is nonzero and points into a direction of improvement.

The second alpha is the ex post alpha given by the deviation from the security market line, which is defined with respect to the true expected returns and covariances of returns, taking the market portfolio as a benchmark. This is the alpha considered in the finance industry and we will use it to study the optimal choice between active and passive investment. We define the (ex post) alpha of asset k by

$$\hat{\alpha}_k := \hat{\mu}_k - R_f - \hat{\beta}^{M,k}(\hat{\mu}^M - R_f),$$

where $\hat{\mu}^M := \sum_k \lambda_k^M \hat{\mu}_k$ is the true expected return of the market portfolio and $\hat{\beta}^{M,k} = \text{COV}(R^k, R^M)/\sigma^2(R^M)$ is the true beta of asset k with respect to the market portfolio. If all investors have homogenous and correct beliefs, i.e. $\mu^i = \hat{\mu}$ and $\text{COV}^i = \text{COV}$ for all i , then all investors hold the market portfolio and $\hat{\alpha}_k = 0$ for all k by Propositions 2.1 and 2.2. Hence, in the standard CAPM with homogenous and correct beliefs there is no portfolio which generates a positive alpha. By contrast, under heterogenous beliefs, there typically exist portfolios generating a positive alpha. To see this recall that in equilibrium

$$\bar{\alpha}_k := \bar{\mu}_k - R_f - \bar{\beta}^{M,k}(\bar{\mu}^M - R_f) = 0$$

for all k by Proposition 2.1. We conclude that if average beliefs differ from the truth ($\bar{\mu} \neq \hat{\mu}$ and/or $\overline{\text{COV}} \neq \text{COV}$), then typically there exists k such that

$\hat{\alpha}_k \neq \bar{\alpha}_k = 0$ and hence there exists a portfolio of risky assets $\bar{\lambda}$, which generates a positive alpha, i.e. $\sum_k \bar{\lambda}_k \hat{\alpha}_k > 0$.

Thus, our CAPM model with heterogenous expectations can explain the existence of a nonzero alpha in equilibrium. However, it turns out that the hunt for alpha opportunities is a zero sum game and that alpha opportunities erode whenever the investor accumulates too much wealth in the economy. Moreover, we will argue that a positive alpha is not necessarily a good criterion for active portfolio management. Hence, our model on the one hand provides a thorough foundation for the alpha and on the other hand casts serious doubt on its use in practical financial decisions.

In order to derive these results we define the ex post or true alpha of investor i 's portfolio as

$$\hat{\alpha}^i := \sum_k \bar{\lambda}_k^i \hat{\alpha}_k,$$

and obtain

$$\begin{aligned} \sum_i w_f^i \hat{\alpha}^i &= \sum_i r^i \left(\sum_j w_f^j \right) \sum_k \bar{\lambda}_k^i \hat{\alpha}_k \\ &= \left(\sum_j w_f^j \right) \sum_k \hat{\alpha}_k \lambda_k^M \\ &= \left(\sum_j w_f^j \right) \left(\hat{\mu}^M - R_f \sum_k \lambda_k^M - (\hat{\mu}^M - R_f) \sum_k \hat{\beta}^{M,k} \lambda_k^M \right) = 0. \end{aligned}$$

Hence, since $w_f^i > 0$ for all i , an investor can generate a positive alpha if and only if there is another investor who generates a negative alpha. We state this zero sum game property in the following proposition.

Proposition 3.1 (The Hunt for Alpha Opportunities is a Zero Sum Game)

In equilibrium

$$\sum_i w_f^i \hat{\alpha}^i = 0.$$

Next, we will address the question, how the alpha of an investor behaves if she accumulates more and more wealth, so that, in the limit, she holds all wealth

in the economy. In practice it has been observed that alpha opportunities erode with the assets under management (Getmansky, 2004, and Agarwal, Daniel, and Naik, 2007). Hence, a fund which becomes too big deprives itself of generating a positive alpha. As we will show, in our model an investor has a zero alpha in the limit, when she has accumulated all the wealth of the economy. The intuition is straightforward: In the limit, an investor who has accumulated all the wealth, must hold the market portfolio which has an alpha of zero. To make this intuition precise, we let $((w_0^{i,n})_i)_n$ be a sequence of wealth profiles. Then, by $\lambda^{M,n}$ we denote the market portfolio under the wealth profile $(w_0^{i,n})_i$, i.e.

$$\lambda_k^{M,n} = \sum_i \bar{\lambda}_k^i r^{i,n}, \quad k = 1, \dots, K,$$

where

$$r^{i,n} = \frac{(1 - \lambda_0^i) w_0^{i,n}}{\sum_j (1 - \lambda_0^j) w_0^{j,n}} \quad \text{for all } i.$$

By $R^{M,n}$ we denote the equilibrium return of the market portfolio under the wealth profile $(w_0^{i,n})_i$, i.e.

$$R^{M,n} = \sum_k \lambda_k^{M,n} R^k.$$

Finally, let $\hat{\mu}^{M,n}$ denote the expectation of $R^{M,n}$ under the true beliefs. Then, for all k , we let $\hat{\alpha}_k^n$ denote the alpha of asset k at the wealth profile $(w_0^{i,n})_i$, i.e.

$$\hat{\alpha}_k^n = \hat{\mu}_k - R_f - \beta_n^{M,k} (\hat{\mu}^{M,n} - R_f),$$

where $\beta_n^{M,k} = \text{COV}(R^k, R^{M,n}) / \sigma^2(R^{M,n})$.

Proposition 3.2 (Erosion of Alpha Opportunities) *Let $((w_0^{i,n})_i)_n$ be a sequence of wealth profiles such that*

$$\lim_{n \rightarrow \infty} \frac{w_0^{i,n}}{\sum_j w_0^{j,n}} = 1$$

for some i . Then

$$\lim_{n \rightarrow \infty} \hat{\alpha}^{i,n} = 0,$$

where

$$\hat{\alpha}^{i,n} = \sum_k \bar{\lambda}_k^i \hat{\alpha}_k^n \quad \text{for all } n.$$

Proof: From $\lim_{n \rightarrow \infty} w_0^{i,n} / \left(\sum_j w_0^{j,n} \right) \rightarrow 1$ it follows that

$$\lim_{n \rightarrow \infty} r^{i,n} = 1,$$

which implies that

$$\lim_{n \rightarrow \infty} \lambda^{M,n} = \lim_{n \rightarrow \infty} \sum_j r^{j,n} \bar{\lambda}^j = \bar{\lambda}^i.$$

Hence,

$$\begin{aligned} \lim_{n \rightarrow \infty} R^{M,n} &= \lim_{n \rightarrow \infty} \sum_k \lambda_k^{M,n} R^k = R^{\bar{\lambda}^i}, \\ \text{and } \lim_{n \rightarrow \infty} \beta_k^{M,n} &= \frac{\text{COV}(R^k, R^{\bar{\lambda}^i})}{\sigma^2(R^{\bar{\lambda}^i})}. \end{aligned}$$

This implies

$$\begin{aligned} \lim_{n \rightarrow \infty} \hat{\alpha}_k^n &= \lim_{n \rightarrow \infty} \left[\hat{\mu}_k - R_f - \frac{\text{COV}(R^k, R^{M,n})}{\sigma^2(R^{M,n})} (\hat{\mu}^{M,n} - R_f) \right] \\ &= \hat{\mu}_k - R_f - \frac{\text{COV}(R^k, R^{\bar{\lambda}^i})}{\sigma^2(R^{\bar{\lambda}^i})} (\hat{\mu}(R^{\bar{\lambda}^i}) - R_f), \end{aligned}$$

where $\hat{\mu}(R^{\bar{\lambda}^i}) = \sum_k \bar{\lambda}_k^i \hat{\mu}_k$. Hence,

$$\lim_{n \rightarrow \infty} \hat{\alpha}^{i,n} = \lim_{n \rightarrow \infty} \sum_k \bar{\lambda}_k^i \hat{\alpha}_k^n = 0$$

as claimed. □

4 Active and Passive Investment

In the previous section we have shown that a CAPM with heterogenous beliefs can explain the existence of a nonzero alpha. We have also seen that the hunt for alpha opportunities is a zero sum game and that an investor, who accumulates too much wealth, deprives himself of generating a positive alpha. The question we are going to address now is much more basic: Is alpha an appropriate performance measure, i.e. should investors base their investment decision on the alpha generated by a fund? In order to answer this question rigorously we have to look at investors' preferences. So the question is, whether an investor's utility is increasing in the

alpha of the portfolio she holds. The main result of this section will answer this question in the negative.

In order to simplify the analysis from now on we will assume that all investors have homogenous and correct beliefs about the covariance matrix of asset returns:

Assumption (HCOV): $\text{COV}^i = \text{COV}$ for all $i = 1, \dots, I$.

This assumption is innocuous as it is sufficient to falsify a hypothesis in a simple model. In our case the hypothesis is that an investor's utility is increasing in the alpha of the portfolio she holds. If this hypothesis is not true in a simple model, where all investors have homogenous beliefs about the covariances of asset returns, then it will not be true in a more general model. Moreover, as we have argued in the introduction, many practitioners do portfolio allocations using historic covariances, so that there is some empirical justification for assuming only heterogeneity in beliefs about expected returns.

In order to analyse the relation between alpha and investors' preferences we consider a particular decision problem, namely the choice between active and passive investment. There is considerable evidence that the share of active investment has been decreasing over time. Cremers and Petajisto (2007) find that between 1983 and 2003 there was a significant decline in the proportion of mutual funds that have a high active share, i.e. even the actively managed funds become more and more passive. A possible reason for this is that actively managed funds typically do not outperform passive investment in a stock market index while at the same time active funds impose high fees. The following analysis will provide a theoretical explanation for the fact that active investment in general does not outperform passive investment. As a consequence, in a stationary economy there will only be passive investment in the long run.

We study the choice between active and passive investment by letting investors choose whether to invest according to an individual belief, which is costly to obtain, or whether to invest according to the average belief, which can be observed

without incurring any costs. More precisely, suppose that each investor i can generate her own belief μ^i about the expected return of the assets. Generating an individual belief reduces investor i 's return by $C^i > 0$. This cost can be interpreted as a cost for information acquisition or as a management fee imposed by an actively managed fund. If the investor does not invest in her own belief she observes the market belief, $\bar{\mu}$, without incurring any costs.

Let $\tilde{\mu}^i \in \{\mu^i, \bar{\mu}\}$ be investor i 's belief. If $\tilde{\mu}^i = \mu^i$, we call i an *active investor* and if $\tilde{\mu}^i = \bar{\mu}$, then i is called a *passive investor*. Recall that under homogenous beliefs about the covariances of asset returns,

$$\bar{\mu} = \sum_i a^i \tilde{\mu}^i, \quad (15)$$

where $a^i = \frac{w_0^i}{\gamma^i} \left(\sum_j \frac{w_0^j}{\gamma^j} \right)^{-1}$. Given the belief $\tilde{\mu}^i$, investor i optimally chooses

$$\lambda^i(\tilde{\mu}^i) := \text{COV}^{-1} \frac{\tilde{\mu}^i - R_f e}{\gamma^i},$$

and invests $1 - \sum_{k=1}^K \lambda_k^i(\tilde{\mu}^i)$ into the riskless asset. Hence, she obtains the portfolio return

$$R(\tilde{\mu}^i) := R_f + \sum_{k=1}^K \lambda_k^i(\tilde{\mu}^i) (R^k - R_f).$$

Clearly, a passive investor will hold the market portfolio λ^M of risky assets.

We assume that investors ex post observe the true expected returns $\hat{\mu}$.⁵ We denote by $U_{\hat{\mu}}^i(\tilde{\mu}^i)$ investor i 's ex post (experienced) utility under the true expected returns if she has invested according to the belief $\tilde{\mu}^i$, i.e.

$$U_{\hat{\mu}}^i(\tilde{\mu}^i) = \mathbb{E}(R(\tilde{\mu}^i)) - \frac{\gamma^i}{2} \sigma^2(R(\tilde{\mu}^i)).$$

Hence,

$$U_{\hat{\mu}}^i(\tilde{\mu}^i) = R_f + \frac{1}{\gamma^i} (\tilde{\mu}^i - R_f e)^T \text{COV}^{-1} (\hat{\mu} - R_f e)$$

⁵The underlying idea is that investors do not revise their investment strategy frequently so that they get enough observations of the asset returns in order to get a very precise estimate of the true expected returns.

$$\begin{aligned}
& -\frac{\gamma^i}{2} \left(\frac{\tilde{\mu}^i - R_f e}{\gamma^i} \right)^T \text{COV}^{-1} \left(\frac{\tilde{\mu}^i - R_f e}{\gamma^i} \right) \\
& = R_f + \frac{1}{\gamma^i} (\tilde{\mu}^i - R_f e)^T \text{COV}^{-1} \left(\hat{\mu} - \frac{1}{2} \tilde{\mu}^i - \frac{1}{2} R_f e \right).
\end{aligned}$$

Observe that $U_{\hat{\mu}}^i(\tilde{\mu}^i)$ is maximized for $\tilde{\mu}^i = \hat{\mu}$, i.e. for the case, where i has correct beliefs. Investor i chooses $\tilde{\mu}^i = \mu^i$ if

$$U_{\hat{\mu}}^i(\mu^i) - C^i \geq U_{\hat{\mu}}^i(\bar{\mu})$$

and $\tilde{\mu}^i = \bar{\mu}$ otherwise.

We define the following scalar product on \mathbb{R}^K :

$$\langle x, y \rangle := x^T \text{COV}^{-1} y, \quad x, y \in \mathbb{R}^K. \quad (16)$$

Observe that $\langle \cdot, \cdot \rangle$ is indeed a scalar product. In particular, $\langle \cdot, \cdot \rangle$ is positive definite since COV and hence COV^{-1} is positive definite. Using $\langle \cdot, \cdot \rangle$ we define the following norm on \mathbb{R}^K :

$$\|x\| := \sqrt{\langle x, x \rangle} = \sqrt{x^T \text{COV}^{-1} x}, \quad x \in \mathbb{R}^K. \quad (17)$$

With respect to this norm, $U_{\hat{\mu}}^i(\mu)$ is decreasing in the distance of μ to $\hat{\mu}$ (the true expectations) as is shown in the following lemma.

Lemma 4.1 *Let $\mu, \mu' \in \mathbb{R}^K$. Then*

$$U_{\hat{\mu}}^i(\mu) - U_{\hat{\mu}}^i(\mu') = \frac{1}{2\gamma^i} (\|\hat{\mu} - \mu'\|^2 - \|\hat{\mu} - \mu\|^2).$$

Hence,

$$U_{\hat{\mu}}^i(\mu) > U_{\hat{\mu}}^i(\mu') \iff \|\hat{\mu} - \mu\| < \|\hat{\mu} - \mu'\|.$$

Proof:

$$U_{\hat{\mu}}^i(\mu) - U_{\hat{\mu}}^i(\mu') = \frac{1}{\gamma^i} \left[\langle \mu - R_f e, \hat{\mu} - \frac{1}{2} \mu - \frac{1}{2} R_f e \rangle \right]$$

$$\begin{aligned}
& - \langle \mu' - R_f e, \hat{\mu} - \frac{1}{2}\mu' - \frac{1}{2}R_f e \rangle \Big] \\
&= \frac{1}{\gamma^i} \left[\langle \mu, \hat{\mu} - \frac{1}{2}\mu \rangle - \langle \mu', \hat{\mu} - \frac{1}{2}\mu' \rangle \right] \\
&= \frac{1}{2\gamma^i} (\|\hat{\mu} - \mu'\|^2 - \|\hat{\mu} - \mu\|^2).
\end{aligned}$$

□

From Lemma 4.1 it follows that investor i chooses $\tilde{\mu}^i = \mu^i$ if and only if

$$\|\bar{\mu} - \hat{\mu}\|^2 - \|\mu^i - \hat{\mu}\|^2 \geq 2C^i\gamma^i. \quad (18)$$

The decision to become active or remain passive thus depends on the accuracy of the average belief, $\|\bar{\mu} - \hat{\mu}\|$, as well as on the accuracy of the investor's belief, $\|\mu^i - \hat{\mu}\|$. We say that $\|\bar{\mu} - \hat{\mu}\|$ measures the “efficiency of the market”, while $\|\mu^i - \hat{\mu}\|$ measures the individual “skill” of investor i . Observe that the more efficient the market is, the smaller the distance of the average belief to the truth. Similarly, the more skilled an investor is, the closer is her belief to the truth. Hence, from (18) it follows that, ceteris paribus, investor i is more inclined to be passive the more risk averse she is, the lower her skill, the higher her investment cost and the more efficient the market is.

Recall that we set out in this paper to answer the question whether alpha is an appropriate performance measure. So the question is, whether the following equivalence holds:

$$\|\bar{\mu} - \hat{\mu}\| \begin{matrix} \geq \\ \leq \end{matrix} \|\mu^i - \hat{\mu}\| \iff \hat{\alpha}^i = \sum_k \bar{\lambda}_k^i \hat{\alpha}_k \begin{matrix} \geq \\ \leq \end{matrix} 0 \quad (19)$$

The following example shows that (19) does not hold in general. More precisely, the example demonstrates that $\hat{\alpha}^i$ can be positive although $\|\bar{\mu} - \hat{\mu}\| < \|\mu^i - \hat{\mu}\|$ so that investor i prefers to be passive at the given belief profile. Conversely, it is possible that $\hat{\alpha}^i$ is negative and $\|\bar{\mu} - \hat{\mu}\| > \|\mu^i - \hat{\mu}\|$ so that investor i prefers to be active if his costs C^i are sufficiently low.

⁶It is without loss of generality to assume that the investor chooses active investment if she is indifferent.

Example 4.1 Let $R_f = 1$ and let there be two risky assets. There are four investors $i = 1, 2, 3, 4$, with the following characteristics:

$$\begin{aligned}\mu^1 &= \begin{pmatrix} 6 \\ 1 \end{pmatrix}, \mu^2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \mu^3 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mu^4 = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\ \gamma^1 &= \gamma^2 = \gamma^3 = \gamma^4 = 2 \\ w_0^1 &= w_0^2 = w_0^3 = w_0^4 = 10\end{aligned}$$

COV is given by

$$\text{COV} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

and the true beliefs are

$$\hat{\mu} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Suppose now that all investors are active. We have $a^1 = a^2 = a^3 = a^4 = 1/4$ and hence

$$\bar{\mu} = a^1 \mu^1 + a^2 \mu^2 + a^3 \mu^3 + a^4 \mu^4 = \begin{pmatrix} 3 \\ \frac{11}{4} \end{pmatrix}.$$

We obtain

$$\begin{aligned}\|\hat{\mu} - \mu^1\|^2 &= \frac{17}{2}, \\ \|\hat{\mu} - \mu^2\|^2 &= \frac{1}{2}, \\ \|\hat{\mu} - \mu^3\|^2 &= \frac{1}{2}, \\ \|\hat{\mu} - \mu^4\|^2 &= 5, \\ \|\hat{\mu} - \bar{\mu}\|^2 &= \frac{25}{32}.\end{aligned}$$

Hence, investors 2 and 3 prefer to be active for sufficiently small costs C^2 , respectively, C^3 , while investors 1 and 4 prefer to be passive for all costs C^1 , respectively C^4 . The optimal portfolios of the investors (everyone is active!) are

$$\lambda^1 = \begin{pmatrix} \frac{5}{4} \\ 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}, \lambda^3 = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix}, \lambda^4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence,

$$\bar{\lambda}^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \bar{\lambda}^2 = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}, \bar{\lambda}^3 = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}, \bar{\lambda}^4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and the market portfolio is⁷

$$\lambda^M = \sum_i r^i \bar{\lambda}^i = \begin{pmatrix} \frac{8}{15} \\ \frac{7}{15} \end{pmatrix},$$

and hence

$$\beta^M = \frac{\text{COV} \lambda^M}{(\lambda^M)^T \text{COV} \lambda^M} = \begin{pmatrix} \frac{110}{113} \\ \frac{105}{113} \end{pmatrix}.$$

This implies

$$\hat{\alpha} = \hat{\mu} - R_f e - \beta^M (\hat{\mu}^M - R_f) = \begin{pmatrix} -\frac{7}{113} \\ \frac{8}{113} \end{pmatrix},$$

from which we compute

$$\begin{aligned} \hat{\alpha}^1 &= \hat{\alpha}^T \bar{\lambda}^1 = -\frac{7}{113}, \\ \hat{\alpha}^2 &= \hat{\alpha}^T \bar{\lambda}^2 = -\frac{2}{113}, \\ \hat{\alpha}^3 &= \hat{\alpha}^T \bar{\lambda}^3 = \frac{3}{113}, \\ \hat{\alpha}^4 &= \hat{\alpha}^T \bar{\lambda}^4 = \frac{8}{113} \end{aligned}$$

Hence, investors 1 and 2 generate a negative alpha by being active, but nevertheless, as we have seen above, investor 2 prefers to be active if her costs C^2 are sufficiently small. Moreover, investors 3 and 4 generate a positive alpha by active investment, but investor 4 prefers to be passive for all costs C^4 .

We are now in the position to define the stability of a CAPM equilibrium under information acquisition. We say that a profile with heterogenous beliefs is stable if no investor wants to deviate from her decision whether to be active or passive:

Definition 4.1 *The profile $\tilde{\mu} = (\tilde{\mu}^1, \dots, \tilde{\mu}^I)$ is **stable**, if the following condition is satisfied: For all i ,*

$$\|\bar{\mu} - \hat{\mu}\|^2 - \|\mu^i - \hat{\mu}\|^2 \geq 2C^i \gamma^i \iff \tilde{\mu}^i = \mu^i. \quad (20)$$

⁷As in the Example 2.1 we choose the exogenous supply of the risky assets, θ^M so that is matches the aggregate demand of the investors.

As our definition of stability makes clear, investors do not take into account that their decision whether to be active or passive may change the average belief $\bar{\mu}$ and the true expected returns $\hat{\mu}$ since it may change the equilibrium price. One objection against our notion of stability might be that we seem to assume that investors know the true expected returns. However, all we require is that investors know how their own skill compares to the efficiency of the market which is something they may have learned from the past.

We will now characterize stable belief profiles. To this end let $\tilde{\mu} = (\tilde{\mu}^1, \dots, \tilde{\mu}^I)$ be some profile of beliefs. Then, from (15) it follows that

$$\bar{\mu} = \left(\sum_{i: \tilde{\mu}^i = \mu^i} a^i \right)^{-1} \sum_{i: \tilde{\mu}^i = \mu^i} a^i \mu^i,$$

whenever $\{i : \tilde{\mu}^i = \mu^i\} \neq \emptyset$, and $\bar{\mu}$ is undetermined, i.e. arbitrary, otherwise.

Proposition 4.1 *There exists no stable profile $\tilde{\mu}$ where some investor is active, i.e. $\tilde{\mu}^i = \mu^i$ for some i .*

Proof: Suppose by way of contradiction that $\tilde{\mu}$ is stable and that $\{i : \tilde{\mu}^i = \mu^i\} \neq \emptyset$. W.l.o.g. let $\{i : \tilde{\mu}^i = \mu^i\} = \{1, \dots, J\}$. Then

$$\bar{\mu} = \frac{1}{\bar{a}_J} \sum_{j=1}^J a^j \mu^j,$$

where $\bar{a}_J := \sum_{j=1}^J a^j$. W.l.o.g. let $\|\mu^1 - \hat{\mu}\| \leq \|\mu^2 - \hat{\mu}\| \leq \dots \leq \|\mu^J - \hat{\mu}\|$. Then

$$\begin{aligned} \|\bar{\mu} - \hat{\mu}\| &= \frac{1}{\bar{a}_J} \left\| \sum_{j=1}^J a^j (\mu^j - \hat{\mu}) \right\| \\ &\leq \frac{1}{\bar{a}_J} \sum_{j=1}^J a^j \|\mu^j - \hat{\mu}\| \\ &\leq \|\mu^J - \hat{\mu}\| \end{aligned}$$

Hence, (20) is violated for $i = J$ contradicting the fact that $\tilde{\mu}$ is stable.

□

Hence, we obtain the paradoxical result that there cannot be active investment in a stable market. The intuition is that the beliefs of active investors determine the average belief so that low-skilled investors prefer to free ride on the better beliefs of high-skilled active investors by investing passively according to the average belief. Proposition 4.1 therefore provides a theoretical explanation for the empirical observation that the share of active investment has been declining constantly over the last twenty years (cf. Cremers and Petajisto, 2007). Clearly, in reality we will always observe active investment as the economy is not stationary. In the language of our model non-stationarity corresponds to a change in the true belief $\hat{\mu}$. If the economy has settled in a stable situation, where there is only passive investment, then a shock to $\hat{\mu}$ may render active investment by a high-skilled investor profitable. Hence, temporarily, we will observe active investment. If then there is no new shock to $\hat{\mu}$ for some period of time, the economy will again settle in a stable situation with passive investment only until the next shock occurs.

Whether or not passive investment indeed leads to a stable situation depends on how $\bar{\mu}$ – which is an arbitrary convention if all investors are passive – relates to the true beliefs $\hat{\mu}$: If the market is very “efficient,” i.e. $\|\bar{\mu} - \hat{\mu}\|$ is close to zero, then (20) is violated for all i , so that every investor being passive ($\tilde{\mu}^i = \bar{\mu}$ for all i) is stable. If, on the contrary, $\|\bar{\mu} - \hat{\mu}\|$ is large, so that there exists an investor i , for whom active investment is profitable, i.e. (18) is satisfied, then passive investment is not stable. In other words, the standard CAPM with homogenous beliefs $\bar{\mu}$ that are close to the true beliefs $\hat{\mu}$ according to the efficiency measure $\|\bar{\mu} - \hat{\mu}\|$, is the only stable outcome of our model.

Proposition 4.2 *The profile $\tilde{\mu} = (\tilde{\mu}^1, \dots, \tilde{\mu}^I)$ is stable if and only if there exists $\bar{\mu}$ such that*

$$(i) \quad \tilde{\mu}^i = \bar{\mu}, \text{ and}$$

$$(ii) \quad \|\bar{\mu} - \hat{\mu}\|^2 < 2C^i\gamma^i + \|\mu^i - \hat{\mu}\|^2,$$

for all i .

Now we are in a position to address the structure of performance fees that are in line with the information acquisition decision of the investors. We have

seen that there cannot be active investment in the long run. In the short run, however, in particular if the true belief $\hat{\mu}$ changes, there is a potential for active investment if the market is inefficient, i.e. $\|\bar{\mu} - \hat{\mu}\|$ large and the skill is high, i.e. $\|\mu^i - \hat{\mu}\|$ is small. Suppose now that an investor cannot invest actively on his own but has to invest into a fund if he wants to be active. This fund sells a portfolio λ which, from the perspective of investor i , corresponds to the belief

$$\mu^i = R_f e + \gamma^i \text{COV} \lambda,$$

which follows from (2). The question then is, how the fee of the fund should look like in order to induce the investor to invest into the fund.

From our previous analysis we obtain two conditions:

- (1) In order to give the fund manager the right incentives, the performance fee should be increasing in the skill of the manager, i.e. decreasing in $\|\mu^i - \hat{\mu}\|$, since $U_{\hat{\mu}}^i(\mu^i)$ is decreasing in $\|\mu^i - \hat{\mu}\|$.
- (2) In order for the investor to become active, the fee must be bounded above by a function that is decreasing in the risk aversion of the investor and in the efficiency of the market.

We get the following result:

Corollary 4.1 *Any performance-fee $C^i = C^i(\|\mu - \hat{\mu}\|, \|\bar{\mu} - \hat{\mu}\|)$, that is decreasing in $\|\mu - \hat{\mu}\|$ and that satisfies*

$$C^i \leq \frac{1}{2\gamma^i} (\|\bar{\mu} - \hat{\mu}\|^2 - \|\mu - \hat{\mu}\|^2),$$

fulfills these conditions.

Hence, the performance fee should reward the skill of the manager but should also discourage the manager to hunt for investment opportunities in efficient markets. Moreover, comparing agents with different degrees of risk aversion, we find that the more risk averse agents have a lower willingness to pay for active portfolio management and therefore are more inclined to be passive.

We have seen that only passive investment is stable. Nevertheless, in the short run, for example, due to changes in the exogenous uncertainty, some investors

may find it profitable to become active. We will now show that active investment is profitable only if the investor's wealth is small relative to the aggregate wealth in the economy. In other words, profitable investment opportunities resulting from inefficient markets (i.e. $\|\bar{\mu} - \hat{\mu}\|$ large) erode if the investor accumulates too much wealth.

Proposition 4.3 *Let $((w_0^{i,n})_i)_n$ be a sequence of wealth profiles such that*

$$\lim_{n \rightarrow \infty} \frac{w_0^{i,n}}{\sum_j w_0^{j,n}} = 1$$

for some i . Then

$$\lim_{n \rightarrow \infty} \|\bar{\mu}^n - \hat{\mu}\| = \|\mu^i - \hat{\mu}\|,$$

where $\bar{\mu}^n = \sum_j a^{j,n} \tilde{\mu}^{j,n}$ with $\tilde{\mu}^{i,n} = \mu^i, \tilde{\mu}^{j,n} \in \{\mu^j, \bar{\mu}^n\}$ for all $j \neq i$, and $a^{j,n} = \frac{w_0^{j,n}}{\gamma^j} \left(\sum_h \frac{w_0^{h,n}}{\gamma^h} \right)^{-1}$ for all j and all n .

Proof: From $\lim_{n \rightarrow \infty} w_0^{i,n} / \left(\sum_j w_0^{j,n} \right) = 1$ it follows that $\lim_{n \rightarrow \infty} w_0^{j,n} / w_0^{i,n} = 0$ for all $j \neq i$. This implies

$$\lim_{n \rightarrow \infty} a^{i,n} = \lim_{n \rightarrow \infty} \frac{1}{\gamma^i} \left(\sum_j \frac{w_0^{j,n}}{w_0^{i,n} \gamma^j} \right)^{-1} = 1.$$

Hence, $\lim_{n \rightarrow \infty} \bar{\mu}^n = \lim_{n \rightarrow \infty} \left(\sum_{j: \tilde{\mu}^{j,n} = \mu^j} a^{j,n} \right)^{-1} \sum_{j: \tilde{\mu}^{j,n} = \mu^j} a^{j,n} \mu^j = \mu^i$ which implies that $\|\bar{\mu}^n - \hat{\mu}\| \rightarrow \|\mu^i - \hat{\mu}\|$. □

5 Conclusion

The main contribution of this paper is twofold. Firstly, our model of a CAPM with heterogenous beliefs provides a general equilibrium foundation for the *alpha* which is heavily used by the finance industry as an indicator for profitable investment opportunities. It turns out that alpha opportunities erode with the assets under management and that the hunt for alpha opportunities is a zero-sum game. Secondly, we have demonstrated that in our model the sign or size of alpha does

not deliver an appropriate criterion for investment decisions. Instead, we have shown that the choice between active and passive investment should be based on a measure of the distance between the individual, respectively average belief and the true expected returns of the assets.

In addition, our paper contributes to the ongoing discussion about the underperformance of active investment by showing that as long as there are active investors in the market, at least one active investor will prefer to become passive. Hence, our model predicts the market share of passive investment to grow over time. This is consistent with the empirical observation that even actively managed funds have become more and more passive over time.⁸

Our model is purely static. In particular, we have assumed that investors have correct expectations about the quality of their beliefs in terms of the distance to the true beliefs. An interesting topic for future research would be to study a dynamic version of our model, where in each period investors can choose between active and passive investment and where they learn about the quality of their beliefs or may even adjust their beliefs over time.

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⁸For a recent study see Cremers and Petajisto (2007).

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